

Renormalization Group Approach to the Beam-Beam Interaction in Circular Colliders

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Abstract

Building on the Renormalization Group (RG) method the beam-beam interaction in circular colliders is studied. A regularized symplectic RG beam-beam map, that describes successfully the long-time asymptotic behavior of the original system has been obtained. The integral of motion possessed by the regularized RG map has been used to construct the invariant phase space density (stationary distribution function), and a coupled set of nonlinear integral equations for the distributions of the two colliding beams has been derived.

1 Introduction

The problem of coherent beam-beam interaction in storage ring colliders is one of the most important, and at the same time one of the most difficult problems in contemporary accelerator physics. Its importance lies in the fact that beam-beam interaction is the basic factor, limiting the luminosity of a circular collider. Nevertheless, some progress in the analytical treatment of the coherent beam-beam interaction has been made [1] - [4], it is still far from being completely understood. In most of the references available the basic trend of analysis follows the perturbative solution of the Vlasov-Poisson equations, where the linearized system is cast in the form of an eigenvalue problem for the eigenmodes.

An important question, which still remains unanswered is how to determine the invariant phase space density (equilibrium distribution function) if such exist. One possible way to approach this problem is to find an integral of motion (at least approximately) under certain conditions. Then the invariant density can be expressed as a generic function of the integral of motion. An attempt in this direction has been made by Alexahin [5], who used the Deprit algorithm to determine the integral of motion (new action variable).

In the present paper we develop a novel approach to the beam-beam interaction in circular colliders, based on the Renormalization Group (RG) method [6]. Originally this method has been proposed as a singular perturbation technique for differential equations. Naive perturbation expansions [7] are well-known to produce secular terms, thus limiting the range of validity of the perturbation solution. The basic idea of the RG method is to remove secular or divergent terms by renormalizing the integration constants of the lowest order perturbative solution. Its extension to discrete symplectic maps is however not straightforward, and should be performed with care. Here we follow the regularization procedure outlined in the paper by Goto and Nozaki [8]. As shown in [8] the naive RG map, obtained as a result of renormalization of the lowest order solution preserves the symplectic symmetry only approximately, and does not describe the long-term behavior of the original map correctly. The symplecticity is recovered by a process of “exponentiation”, yielding a symplectic RG map together with an explicit expression for the nonlinear tune shift. An alternative version of the RG method, based on the envelope technique [9] has been applied to study non symplectic maps.

The paper is organized as follows. In the next Section we derive the one-dimensional nonlinear beam-beam map. In Section 3 the regularized RG map and its integral of motion are obtained. The integral of motion thus found is further used in Section 4 to derive a set of coupled integral equations of Haissinski type for the invariant phase space density.

2 The Nonlinear Beam-Beam Map

We begin with the one-dimensional model of coherent beam-beam interaction in the vertical (q) direction, described by the Hamiltonian

$$H_k = \frac{\dot{\chi}_k}{2}(p^2 + q^2) + \lambda_k \delta_p(\theta) V_k(q; \theta), \quad (2.1)$$

where the normalized beam-beam potential $V_k(q; \theta)$ satisfies the Poisson equation

$$\frac{\partial^2 V_k}{\partial q^2} = 4\pi \int_{-\infty}^{\infty} dp f_{3-k}(q, p; \theta), \quad (2.2)$$

and

$$\lambda_k = \frac{Rr_e N_{3-k} \beta_{kq}^*}{\gamma_{k0} L_{(3-k)x}} \frac{1 + \beta_{k0} \beta_{(3-k)0}}{\beta_{k0}^2} \simeq \frac{2Rr_e N_{3-k} \beta_{kq}^*}{\gamma_{k0} L_{(3-k)x}}. \quad (2.3)$$

Here, $(k = 1, 2)$ labels the counter-propagating beams, θ is the azimuthal angle, $\dot{\chi}_k = R\beta_{kq}^{-1}$ is the derivative of the phase advance with respect to θ , R is the mean machine radius, r_e is the classical electron radius, $N_{1,2}$ is the total number of particles in either beam, β_{kq}^* is the vertical beta-function at the interaction point, and L_{kx} is the horizontal dimension of the beam ribbon. In addition, the distribution function $f_k(q, p; \theta)$ is a solution to the Vlasov equation

$$\frac{\partial f_k}{\partial \theta} + \dot{\chi}_k p \frac{\partial f_k}{\partial q} - \frac{\partial H_k}{\partial q} \frac{\partial f_k}{\partial p} = 0. \quad (2.4)$$

In order to build the iterative beam-beam map we formally solve the Hamilton's equations of motion

$$\dot{q} = \frac{dq}{d\theta} = \dot{\chi}_k p, \quad \dot{p} = \frac{dp}{d\theta} = -\dot{\chi}_k q - \lambda_k \delta_p(\theta) V'_k(q; \theta), \quad (2.5)$$

where the prime implies differentiation with respect to the spatial variable q . By defining the state vector

$$\mathbf{z} = \begin{pmatrix} q \\ p \end{pmatrix}, \quad (2.6)$$

we can rewrite Eq. (2.5) in a vector form

$$\dot{\mathbf{z}} = \widehat{\mathbf{K}}(\theta) \mathbf{z} + \mathbf{F}(\mathbf{z}; \theta), \quad (2.7)$$

where

$$\widehat{\mathbf{K}}(\theta) = \begin{pmatrix} 0 & \dot{\chi}_k \\ -\dot{\chi}_k & 0 \end{pmatrix}, \quad \mathbf{F}(\mathbf{z}; \theta) = \begin{pmatrix} 0 \\ -\lambda_k \delta_p(\theta) V'_k(q; \theta) \end{pmatrix}. \quad (2.8)$$

Performing a linear transformation defined as

$$\mathbf{z} = \widehat{\mathcal{M}}(\theta) \vec{\xi}, \quad (2.9)$$

where the matrix $\widehat{\mathcal{M}}$ is a solution of the linear equation with a supplementary initial condition

$$\dot{\widehat{\mathcal{M}}} = \widehat{\mathbf{K}}(\theta) \widehat{\mathcal{M}}, \quad \widehat{\mathcal{M}}(\theta_0) = \widehat{\mathbf{I}}, \quad (2.10)$$

we write the equation for the transformed state vector $\vec{\xi}$ as follows:

$$\vec{\xi} = \widehat{\mathcal{M}}^{-1}(\theta) \mathbf{F}(\mathbf{z}; \theta), \quad \vec{\xi}(\theta_0) = \mathbf{z}_0. \quad (2.11)$$

Equation (2.11) can be solved directly to give

$$\mathbf{z}(\theta) = \widehat{\mathcal{M}}(\theta) \mathbf{z}_0 + \int_{\theta_0}^{\theta} d\tau \widehat{\mathcal{M}}(\theta) \widehat{\mathcal{M}}^{-1}(\tau) \mathbf{F}(\mathbf{z}(\tau); \tau). \quad (2.12)$$

It can be easily checked that the matrix of fundamental solutions $\widehat{\mathcal{M}}$ to the unperturbed problem is of the form

$$\widehat{\mathcal{M}}(\theta) = \begin{pmatrix} \cos [\chi_k(\theta) - \chi_k(\theta_0)] & \sin [\chi_k(\theta) - \chi_k(\theta_0)] \\ -\sin [\chi_k(\theta) - \chi_k(\theta_0)] & \cos [\chi_k(\theta) - \chi_k(\theta_0)] \end{pmatrix}, \quad (2.13)$$

so that

$$\mathbf{z}(\theta) = \widehat{\mathcal{M}}(\theta) \mathbf{z}_0 - \lambda_k \int_{\theta_0}^{\theta} d\tau \delta_p(\tau) V'_k(q(\tau); \tau) \begin{pmatrix} \sin [\chi_k(\theta) - \chi_k(\tau)] \\ \cos [\chi_k(\theta) - \chi_k(\tau)] \end{pmatrix}. \quad (2.14)$$

Applying the above expression (2.14) in a small ε -interval $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ around the interaction point (located at θ_0) and then taking the limit $\varepsilon \rightarrow 0$ we obtain the kick map

$$q_{\text{kick}} = q_0, \quad p_{\text{kick}} = p_0 - \lambda_k V'_k(q_0). \quad (2.15)$$

In order to obtain the rotation map in between successive kicks we apply once again expression (2.14) in the interval $\theta \in (\theta_0 + \varepsilon, \theta_0 + 2\pi - \varepsilon)$

$$\begin{aligned} q_{\text{rot}} &= q_{\text{kick}} \cos 2\pi\nu_k + p_{\text{kick}} \sin 2\pi\nu_k, \\ p_{\text{rot}} &= -q_{\text{kick}} \sin 2\pi\nu_k + p_{\text{kick}} \cos 2\pi\nu_k. \end{aligned} \quad (2.16)$$

Combining Eqs. (2.15) and (2.16) we finally arrive at the one-turn beam-beam map

$$\begin{aligned} q_{n+1} &= q_n \cos 2\pi\nu_k + [p_n - \lambda_k V'_k(q_n)] \sin 2\pi\nu_k, \\ p_{n+1} &= -q_n \sin 2\pi\nu_k + [p_n - \lambda_k V'_k(q_n)] \cos 2\pi\nu_k. \end{aligned} \quad (2.17)$$

It is important to note that the one-turn beam-beam map (2.17) is symplectic, since its Jacobian determinant is equal to unity

$$\det \frac{\partial(q_{n+1}, p_{n+1})}{\partial(q_n, p_n)} = \det \begin{pmatrix} \cos 2\pi\nu_k - \lambda_k V''_k \sin 2\pi\nu_k & \sin 2\pi\nu_k \\ -\sin 2\pi\nu_k - \lambda_k V''_k \cos 2\pi\nu_k & \cos 2\pi\nu_k \end{pmatrix} \equiv 1. \quad (2.18)$$

3 Renormalization Group Reduction of the Beam-Beam Map

The one-turn beam-beam map, derived in the previous Section can be further simplified by eliminating the canonical momentum variable p from (2.17). Multiplying the first of Eqs. (2.17) by $\cos 2\pi\nu_k$, multiplying the second one by $-\sin 2\pi\nu_k$, and summing the two equations up we find

$$q_{n+1} \cos \omega_k - p_{n+1} \sin \omega_k = q_n, \quad (3.1)$$

where

$$\omega_k = 2\pi\nu_k. \quad (3.2)$$

Using Eq. (3.1) we obtain a second order difference equation

$$\hat{\mathcal{L}}q_n = q_{n+1} - 2q_n \cos \omega_k + q_{n-1} = -\epsilon\lambda_k V'_k(q_n) \sin \omega_k, \quad (3.3)$$

where ϵ is a formal small parameter (set to unity at the end of the calculations), taking into account the fact that the beam-beam kick is small and can be treated as perturbation.

Next we consider an asymptotic solution of the map (3.3) for small ϵ by means of the RG method. The naive perturbation expansion

$$q_n = q_n^{(0)} + \epsilon q_n^{(1)} + \epsilon^2 q_n^{(2)} + \dots \quad (3.4)$$

when substituted into Eq. (3.3) yields the perturbation equations order by order

$$\hat{\mathcal{L}}q_n^{(0)} = 0, \quad (3.5)$$

$$\hat{\mathcal{L}}q_n^{(1)} = -\lambda_k V'_k(q_n^{(0)}) \sin \omega_k, \quad (3.6)$$

$$\hat{\mathcal{L}}q_n^{(2)} = -\lambda_k q_n^{(1)} V''_k(q_n^{(0)}) \sin \omega_k, \quad (3.7)$$

$$\hat{\mathcal{L}}q_n^{(3)} = -\lambda_k \left[\frac{q_n^{(1)2}}{2} V'''_k(q_n^{(0)}) + q_n^{(2)} V''_k(q_n^{(0)}) \right] \sin \omega_k, \quad (3.8)$$

Solving Eq. (3.5) for the zeroth order contribution we obtain the obvious result

$$q_n^{(0)} = A_k e^{i\omega_k n} + \text{c.c.} = 2|A_k| \cos(\omega_k n + \phi_k), \quad (3.9)$$

$$p_n^{(0)} = iA_k e^{i\omega_k n} + \text{c.c.} = -2|A_k| \sin(\omega_k n + \phi_k), \quad (3.10)$$

where A_k is a complex integration constant, whose amplitude and phase are $|A_k|$ and ϕ_k respectively.

Let us assume for the time being that the beam-beam potential $V_k(q)$ is a known function of the vertical displacement q . In what follows it will prove efficient to take into account the fact that the beam-beam potential $V_k(q)$ is an even function of the coordinate q . Odd multipole contributions to $V_k(q)$ will give rise to a shift in the closed orbit, and can be easily incorporated in the the calculations presented below. It is straightforward to check that the Fourier image of the beam-beam potential $V_k(\lambda)$, defined as

$$V_k(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda V_k(\lambda) e^{i\lambda q}, \quad V_k(\lambda) = \int_{-\infty}^{\infty} dq V_k(q) e^{-i\lambda q}, \quad (3.11)$$

retains the symmetry properties of $V_k(q)$, that is:

$$V_k(-\lambda) = V_k(\lambda). \quad (3.12)$$

Using the expansion [10, 11]

$$e^{iz \cos \varphi} = \sum_{m=-\infty}^{\infty} i^m \mathcal{J}_m(z) e^{im\varphi}, \quad (3.13)$$

where $\mathcal{J}_m(z)$ is the Bessel function of the first kind of order m , and the explicit form of the zero order solution (3.9) we find

$$V'_k(q_n^{(0)}) = \sum_{M=1}^{\infty} \mathcal{C}_k^{(M)} A_k^{2M-1} e^{i(2M-1)\omega_k n} + \text{c.c.} \quad (3.14)$$

Here the coefficients $\mathcal{C}_k^{(M)}$ are functions of the amplitude $|A_k|$ and are given by the expression

$$\mathcal{C}_k^{(M)}(|A_k|) = \frac{1}{\pi} \frac{(-1)^M}{|A_k|^{2M-1}} \int_0^{\infty} d\lambda \lambda V_k(\lambda) \mathcal{J}_{2M-1}(2\lambda|A_k|). \quad (3.15)$$

Similarly for the second derivative of the beam-beam potential $V''_k(q_n^{(0)})$, entering the second order perturbation equation (3.7) we have

$$V''_k(q_n^{(0)}) = \mathcal{D}_k^{(0)} + \sum_{M=1}^{\infty} \mathcal{D}_k^{(M)} A_k^{2M} e^{i2M\omega_k n} + \text{c.c.}, \quad (3.16)$$

where

$$\mathcal{D}_k^{(0)}(|A_k|) = -\frac{1}{\pi} \int_0^{\infty} d\lambda \lambda^2 V_k(\lambda) \mathcal{J}_0(2\lambda|A_k|), \quad (3.17)$$

$$\mathcal{D}_k^{(M)}(|A_k|) = \frac{1}{\pi} \frac{(-1)^{M+1}}{|A_k|^{2M}} \int_0^{\infty} d\lambda \lambda^2 V_k(\lambda) \mathcal{J}_{2M}(2\lambda|A_k|). \quad (3.18)$$

From the recursion property of Bessel functions [10, 11]

$$\mathcal{J}_{\nu-1}(z) + \mathcal{J}_{\nu+1}(z) = \frac{2\nu}{z} \mathcal{J}_{\nu}(z) \quad (3.19)$$

we deduce an important relation to be used later

$$\mathcal{D}_k^{(N)} - \mathcal{D}_k^{(N+1)} |A_k|^2 = (2N+1) \mathcal{C}_k^{(N+1)}. \quad (3.20)$$

The solutions of the perturbation equations (3.6) and (3.7), taking into account (3.20) are given by

$$\begin{aligned} q_n^{(1)} &= \frac{i\lambda_k n}{2} \mathcal{C}_k^{(1)} A_k e^{i\omega_k n} + \frac{\lambda_k \sin \omega_k}{2} \sum_{M=1}^{\infty} \tilde{\mathcal{C}}_k^{(M+1)} A_k^{2M+1} e^{i(2M+1)\omega_k n} + \text{c.c.}, \quad (3.21) \\ q_n^{(2)} &= -\frac{\lambda_k^2}{8} \mathcal{C}_k^{(1)2} (n^2 + in \cot \omega_k) A_k e^{i\omega_k n} + \frac{i\lambda_k^2 \sin \omega_k}{4} n \sum_{N=1}^{\infty} \tilde{\mathcal{C}}_k^{(N+1)} \mathcal{D}_k^{(N)} |A_k|^{4N} A_k e^{i\omega_k n} \\ &\quad + \frac{i\lambda_k^2 \sin \omega_k}{4} \mathcal{C}_k^{(1)} \sum_{N=1}^{\infty} (2N+1) \tilde{\mathcal{C}}_k^{(N+1)} \left[n + i \frac{\sin(2N+1)\omega_k}{\cos \omega_k - \cos(2N+1)\omega_k} \right] A_k^{2N+1} e^{i(2N+1)\omega_k n} \\ &\quad + \frac{\lambda_k^2 \sin^2 \omega_k}{4} \mathcal{D}_k^{(0)} \sum_{N=1}^{\infty} \frac{\tilde{\mathcal{C}}_k^{(N+1)}}{\cos \omega_k - \cos(2N+1)\omega_k} A_k^{2N+1} e^{i(2N+1)\omega_k n} \\ &\quad + \frac{\lambda_k^2 \sin^2 \omega_k}{4} \sum_{M=1}^{\infty} \sum_{N=1}^{\infty} \frac{\tilde{\mathcal{C}}_k^{(M+1)} \mathcal{D}_k^{(N)}}{\cos \omega_k - \cos[2(M+N)+1]\omega_k} A_k^{2(M+N)+1} e^{i[2(M+N)+1]\omega_k n} \\ &\quad + \frac{\lambda_k^2 \sin^2 \omega_k}{4} \sum_{M=1}^{\infty} \sum_{N=1}^{\infty} \frac{\tilde{\mathcal{C}}_k^{(M+1)} \mathcal{D}_k^{(N)} |A_k|^{4N}}{\cos \omega_k - \cos[2(M-N)+1]\omega_k} A_k^{2(M-N)+1} e^{i[2(M-N)+1]\omega_k n} + \text{c.c.} \quad (3.22) \end{aligned}$$

where the summation in the last term of Eq. (3.22) is performed for $M \neq N$, and

$$\tilde{\mathcal{C}}_k^{(N+1)} = \frac{\mathcal{C}_k^{(N+1)}}{\cos \omega_k - \cos(2N+1)\omega_k}. \quad (3.23)$$

To remove secular terms, proportional to n and n^2 we define the renormalization transformation $A_k \rightarrow \tilde{A}_k(n)$ by collecting all terms proportional to the fundamental harmonic $e^{i\omega_k n}$

$$\tilde{A}_k(n) = A_k + \epsilon \frac{i\lambda_k n}{2} \mathcal{C}_k^{(1)} A_k$$

$$+\epsilon^2 \left[-\frac{\lambda_k^2}{8} \mathcal{C}_k^{(1)2} (n^2 + in \cot \omega_k) + \frac{i\lambda_k^2 \sin \omega_k}{4} n \sum_{N=1}^{\infty} \tilde{\mathcal{C}}_k^{(N+1)} \mathcal{D}_k^{(N)} |A_k|^{4N} \right] A_k. \quad (3.24)$$

Solving perturbatively Eq. (3.24) for A_k in terms of $\tilde{A}_k(n)$ we obtain

$$A_k = \left[1 - \epsilon \frac{i\lambda_k n}{2} \mathcal{C}_k^{(1)} + O(\epsilon^2) \right] \tilde{A}_k(n). \quad (3.25)$$

A discrete version of the RG equation can be defined by considering the difference

$$\begin{aligned} \tilde{A}_k(n+1) - \tilde{A}_k(n) &= \epsilon \frac{i\lambda_k}{2} \mathcal{C}_k^{(1)} A_k \\ &+ \epsilon^2 \left[-\frac{\lambda_k^2}{8} \mathcal{C}_k^{(1)2} (2n+1 + i \cot \omega_k) + \frac{i\lambda_k^2 \sin \omega_k}{4} \sum_{N=1}^{\infty} \tilde{\mathcal{C}}_k^{(N+1)} \mathcal{D}_k^{(N)} |A_k|^{4N} \right] A_k. \end{aligned} \quad (3.26)$$

Substituting the expression for A_k in terms of $\tilde{A}_k(n)$ [see Eq. (3.25)] into the above Eq. (3.26) we can eliminate the secular terms up to $O(\epsilon^2)$. The result is

$$\begin{aligned} \tilde{A}_k(n+1) &= \left[1 + \epsilon \frac{i\lambda_k}{2} \mathcal{C}_k^{(1)} - \epsilon^2 \frac{\lambda_k^2}{8} \mathcal{C}_k^{(1)2} (1 + i \cot \omega_k) \right. \\ &\quad \left. + i\epsilon^2 \frac{\lambda_k^2 \sin \omega_k}{4} \sum_{N=1}^{\infty} \tilde{\mathcal{C}}_k^{(N+1)} \mathcal{D}_k^{(N)} |\tilde{A}_k(n)|^{4N} \right] \tilde{A}_k(n). \end{aligned} \quad (3.27)$$

This naive RG map does not preserve the symplectic symmetry and does not have a *constant of motion*. To recover the symplectic symmetry we regularize the naive RG map by noting that the coefficient in the square brackets, multiplying $\tilde{A}_k(n)$ can be exponentiated:

$$\tilde{A}_k(n+1) = \tilde{A}_k(n) \exp \left[i\tilde{\omega}_k \left(|\tilde{A}_k(n)| \right) \right], \quad (3.28)$$

where

$$\tilde{\omega}_k \left(|\tilde{A}_k(n)| \right) = \epsilon \frac{\lambda_k \mathcal{C}_k^{(1)}}{2} + \epsilon^2 \frac{\lambda_k^2}{8} \left(-\mathcal{C}_k^{(1)2} \cot \omega_k + 2 \sin \omega_k \sum_{N=1}^{\infty} \tilde{\mathcal{C}}_k^{(N+1)} \mathcal{D}_k^{(N)} |\tilde{A}_k(n)|^{4N} \right). \quad (3.29)$$

It is clear now that the regularized RG map (3.28) possesses the obvious integral of motion:

$$|\tilde{A}_k(n+1)| = |\tilde{A}_k(n)| = \sqrt{\frac{J_k}{2}}. \quad (3.30)$$

It is worthwhile to note that the secular coefficients of the $(2N+1)$ -st harmonic $e^{i(2N+1)\omega_k n}$ can be summed up to give a renormalized coefficient, which expressed in terms of $\tilde{A}_k(n)$ does not contain secular terms.

Proceeding in the same way as above, we can write the canonical conjugate momentum p_n in the form

$$p_n = i\tilde{B}_k(n)e^{i\omega_k n} + \text{c.c.} + \text{higher harmonics}, \quad (3.31)$$

where

$$\tilde{B}_k(n+1) = \tilde{B}_k(n)\exp\left[i\tilde{\omega}_k\left(|\tilde{A}_k(n)|\right)\right]. \quad (3.32)$$

Using now the relation (3.1) between the canonical conjugate variables (q, p) we can express the renormalized amplitude $\tilde{B}_k(n)$ in terms of $\tilde{A}_k(n)$ as

$$\tilde{B}_k(n) = i\frac{e^{-i(\omega_k + \tilde{\omega}_k)} - \cos \omega_k}{\sin \omega_k} \tilde{A}_k(n). \quad (3.33)$$

Neglecting higher harmonics and iterating Eqs. (3.28) and (3.32) we can write the renormalized solution of the beam-beam map (2.17)

$$q_n = \sqrt{2J_k \cos \psi_k(J_k; n)}, \quad (3.34)$$

$$p_n = \alpha_k(J_k)\sqrt{2J_k \cos \psi_k(J_k; n)} - \beta_k(J_k)\sqrt{2J_k \sin \psi_k(J_k; n)}, \quad (3.35)$$

where

$$\psi_k(J_k; n) = [\omega_k + \tilde{\omega}_k(J_k)]n + \tilde{\phi}_k, \quad (3.36)$$

$$\alpha_k(J_k) = \frac{\cos \omega_k - \cos [\omega_k + \tilde{\omega}_k(J_k)]}{\sin \omega_k}, \quad \beta_k(J_k) = \frac{\sin [\omega_k + \tilde{\omega}_k(J_k)]}{\sin \omega_k}. \quad (3.37)$$

It is easy to see that the integral of motion J_k has the form of a generalized Courant-Snyder invariant and can be written as

$$2J_k = q^2 + \frac{[p - \alpha_k(J_k)q]^2}{\beta_k^2(J_k)}. \quad (3.38)$$

It is important to emphasize that Eq. (3.38) comprises a transcendental equation for the invariant J_k as a function of the canonical variables (q, p) , since the coefficients α_k and β_k depend on J_k themselves.

4 The Invariant Phase Space Density

If an integral of motion J_k of the beam-beam map (2.17) exists, it can be proved that the invariant phase space density $f_k^{(I)}(q, p)$ [which is a solution to the Vlasov equation (2.4)] is a generic function of J_k , that is

$$f_k^{(I)}(q, p) = F_k(J_k) \quad (k = 1, 2). \quad (4.1)$$

Here $F_k(z)$ is a generic function of its argument. Since the integral of motion J_k is a functional of the invariant density of the opposing beam $f_{3-k}^{(I)}(q, p)$, Eq. (4.1) comprises a coupled system of nonlinear integral equations for the invariant densities of the two counter-propagating beams. Let us find the integral of motion [see Eq. (3.38)] up to first order in the perturbation parameter ϵ . We have

$$J_k = J_0 - \frac{\lambda_k \mathcal{C}_k^{(1)}(J_0)}{2} (p^2 \cot \omega_k + pq), \quad (4.2)$$

where

$$J_0 = \frac{1}{2} (p^2 + q^2). \quad (4.3)$$

The Fourier image of the beam-beam potential

$$V_k(\lambda) = -\frac{4\pi}{\lambda^2} \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dp' f_{3-k}^{(I)}(q', p') \cos \lambda q', \quad (4.4)$$

obtained by solving the Poisson equation (2.2) is next substituted into the corresponding expression [see Eq. (3.15)] for the coefficient $\mathcal{C}_k^{(1)}(J_0)$. Taking into account the recursion relation (3.19) as well as the identities [10]

$$\int_0^{\infty} dx \mathcal{J}_0(x) \cos ax = \frac{1}{\sqrt{1-a^2}} \quad [0 < a < 1], \quad (4.5)$$

$$\int_0^{\infty} dx \mathcal{J}_2(x) \cos ax = -\frac{2a^2 - 1}{\sqrt{1-a^2}} \quad [0 < a < 1] \quad (4.6)$$

we obtain

$$\mathcal{C}_k^{(1)}(J_0) = \frac{8}{J_0} \int_{-\infty}^{\infty} dp' \int_0^{\sqrt{2J_0}} dq' f_{3-k}^{(I)}(q', p') \sqrt{2J_0 - q'^2}. \quad (4.7)$$

Thus, we finally arrive at the system of integral equations for the invariant phase space densities $f_k^{(I)}(q, p)$

$$f_1^{(I)}(q, p) = C_1 F_1 \left[J_0 - \frac{4\lambda_1}{J_0} (p^2 \cot \omega_1 + pq) \int_{-\infty}^{\infty} dp' \int_0^{\sqrt{2J_0}} dq' f_2^{(I)}(q', p') \sqrt{2J_0 - q'^2} \right], \quad (4.8)$$

$$f_2^{(I)}(q, p) = C_2 F_2 \left[J_0 - \frac{4\lambda_2}{J_0} (p^2 \cot \omega_2 + pq) \int_{-\infty}^{\infty} dp' \int_0^{\sqrt{2J_0}} dq' f_1^{(I)}(q', p') \sqrt{2J_0 - q'^2} \right], \quad (4.9)$$

where

$$C_k = \left[\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq F_k(q, p) \right]^{-1}. \quad (4.10)$$

It is instructive to calculate the first order nonlinear incoherent beam-beam tune shift. According to Eq. (3.29) we have

$$\tilde{\omega}_k^{(1)}(J_k) = \frac{4\lambda_k}{J_k} \int_{-\infty}^{\infty} dp' \int_0^{\sqrt{2J_k}} dq' f_{3-k}^{(I)}(q', p') \sqrt{2J_k - q'^2}. \quad (4.11)$$

Since we are interested in the first order $O(\lambda_k)$ contribution, we substitute in Eq. (4.11) the unperturbed phase space density

$$f_k^{(I)}(q, p) = \frac{1}{2\pi\sigma_k^2} \exp\left(-\frac{p^2 + q^2}{2\sigma_k^2}\right). \quad (4.12)$$

Simple manipulations yield

$$\tilde{\omega}_k^{(1)}(J_k) = \frac{2\lambda_k}{\sigma_{3-k}\sqrt{2\pi}} \exp\left(-\frac{J_k}{2\sigma_{3-k}^2}\right) \int_0^{\pi} d\psi (1 + \cos\psi) \exp\left(\frac{J_k}{2\sigma_{3-k}^2} \cos\psi\right). \quad (4.13)$$

Taking into account the integral representation of the modified Bessel function $\mathcal{I}_n(z)$ (see e.g. Ref. [11])

$$\mathcal{I}_n(z) = \frac{1}{\pi} \int_0^{\pi} d\tau \cos(n\tau) e^{z \cos\tau}, \quad (4.14)$$

we obtain

$$\tilde{\omega}_k^{(1)}(J_k) = \frac{\lambda_k \sqrt{2\pi}}{\sigma_{3-k}} \exp\left(-\frac{J_k}{2\sigma_{3-k}^2}\right) \left[\mathcal{I}_0\left(\frac{J_k}{2\sigma_{3-k}^2}\right) + \mathcal{I}_1\left(\frac{J_k}{2\sigma_{3-k}^2}\right) \right]. \quad (4.15)$$

A similar expression for the incoherent beam-beam tune shift was obtained in [3].

5 Concluding Remarks

As a result of the investigation performed we have obtained a regularized symplectic RG beam-beam map, that describes correctly the long-time asymptotic behavior of the original system. It has been shown that the regularized RG map possesses an integral of motion, which can be computed to any desired order. The invariant phase space density (stationary distribution function) has been constructed as a generic function of the integral of motion, and a coupled set of nonlinear integral equations for the distributions of the two colliding beams has been derived. Based on the explicit form of the regularized RG map, the incoherent beam-beam tune shift has been computed to first order in the beam-beam parameter.

It is worthwhile to note that the method presented here is also applicable to study the four-dimensional symplectic beam-beam map, governing the dynamics of counter-propagating beams in the plane transverse to the particle orbit.

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